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Stochastic stability of non-gyroscopic viscoelastic systems

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Abstract

In this paper the dynamic stability of non-gyroscopic viscoelastic systems under multiple parametric excitations is investigated. The largest Lyapunov exponent as an indicator of the almost-sure asymptotic stability of the system is obtained by applying the stochastic averaging method together with Khasminskii's technique. The integral term arising from the viscoelastic effect is averaged by making use of Larianov's method. As an application, the flexural–torsional instability of a deep rectangular viscoelastic beam under stochastically fluctuating central load and end moments applied simultaneously is investigated. Both cases of follower and non-follower central fluctuating load are included in this analysis.

Keywords: Viscoelastic; Ergodic; Itô equations; Lyapunov exponent

1. Introduction

The dynamic stability of non-gyroscopic conservative elastic systems such as beams and columns under non-follower force, and of non-gyroscopic, non-conservative systems such as plates in supersonic flow under deterministic axial thrust have been treated in detail by Bolotin (1963, 1964). Applications to the dynamic stability of structures under periodic forces may also be found in the articles by Mettler (1966, 1968). The dynamic stability problem of elastic beams under deterministic parametric and external loads associated with different end conditions has been investigated by many researchers, such as Saito and Koizumi (1982), Huang and Hung (1984).

In some engineering applications, there exist situations in which the exciting forces cannot be described adequately in the form of deterministic functions alone and a modelling based on probabilistic terms is needed. Some examples of such stochastic excitation are forces generated by jet and rocket engines in modern high powered aircraft and missile structures, excitation due to earthquakes, ocean waves, and wind gusts. Furthermore, even when the excitation can be described to be principally deterministic, it may be more realistic to investigate the stability of the system by subjecting it to an additional random perturbation. To investigate the stability of linear stochastic systems, Khasminskii (1967) developed a technique,

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based on the concept of Lyapunov exponent as an almost-sure stability indicator, by studying the dynamical stability of linear systems described by Itô stochastic differential equations whose solutions are Markov processes.

The stochastic stability of certain two-dimensional systems using Khasminskii's technique (1967), has been investigated by Mitchell and Kozin (1974). By using the method of stochastic averaging developed by Stratonovich (1963) and Khasminskii (1966), the moment stability of non-gyroscopic elastic systems under random loading was examined by Ariaratnam and Srikantaiah (1978). The sample stability of the same class of problems under white noise excitation was also studied by Ariaratnam et al. (1990) and under real noise excitation by Ariaratnam et al. (1992) by using a combination of the stochastic averaging method and Khasminskii's technique.

A two-dimensional viscoelastic system under a phase modulated bounded noise process was studied by Ariaratnam (1995). Potapov and Bonder (1996) investigated the vibrations of elastic and viscoelastic plates under random loading and obtained stability conditions in the mean and mean square sense. Ariaratnam and Abdelrahman (2001) studied the stability of viscoelastic plates in supersonic gas flow and under stochastic axial thrust and obtained explicit expressions for the largest Lyapunov exponent.

In the present paper, the stability of non-gyroscopic viscoelastic systems subjected to multiple parametric random excitations described by a linear combination of ergodic stochastic processes of small intensity and short correlation time is investigated. The motivation for the study of this class of problems stems from the investigation of flexural–torsional instability of a deep rectangular viscoelastic beam subjected to stochastically fluctuating central transverse load and end moments applied simultaneously. The almost-sure stability conditions are obtained by using a combination of the method of stochastic averaging and Khasminskii's technique together with Larianov's method (1969) for averaging the viscoelastic terms. In analogy with the deterministic results obtained by Mettler (1968), it is found in this analysis that only those values of the excitation spectrum near twice the system natural frequencies and the sum and difference of the natural frequencies influence the stability in the first approximation. As an application, the flexural–torsional instability of a deep rectangular viscoelastic beam under follower or non-follower central transverse loads and end moments is investigated in the present analysis.

2. Formulation

Consider a dynamical system described by the following non-dimensional linear stochastic differential equations:

$$\ddot{q}_i + \omega_i^2 q_i - \omega_i^2 \sum_{j=1}^{n} e_{ij}\mathbf{R}[q_j] + 2\sum_{j=1}^{n} \beta_{ij}\dot{q}_j + \omega_i \sum_{j=1}^{n} k_{ij}\eta_{ij}(t) q_j = 0, \quad i = 1,2,\ldots,n \tag{1}$$

where q_i are the non-dimensional generalized coordinates, ω_i are the non-dimensional natural frequencies, e_{ij} are non-dimensional constant coefficients and $\mathbf{R}$ is a viscoelastic relaxation operator given by

$$\mathbf{R}[\psi(t)] = \int_0^t R(t-\tau)\psi(\tau)\,\mathrm{d}\tau \tag{2}$$

The non-dimensional coefficients terms β_{ij} and k_{ij} are the small viscous damping and stiffness coefficients, respectively. The non-dimensional processes $\eta_{ij}(t)$, $i,j = 1,2,\ldots,n$, denote the multiple parametric excitations and are described as a linear combination of the non-dimensional ergodic stochastic processes $\xi_m(t)$, where $m = 1,2,\ldots,N$. The processes $\xi_m(t)$ are considered to have zero mean and a sufficiently small correlation time:

$$\eta_{ij}(t) = \sum_{m=1}^{N} c_{ijm}\xi_m(t) \quad i,j = 1,2,\ldots,n \tag{3}$$

where c_{ijm} are non-dimensional constant coefficients giving the contribution of the ergodic stochastic processes $\xi_m(t)$, $m = 1,2,\ldots,N$, to the multiplicative processes $\eta_{ij}(t)$, $i,j = 1,2,\ldots,n$. The system of equations (1) describes exactly the parametrically excited motion of non-gyroscopic, discrete, linear mechanical systems with n-degrees of freedom about the equilibrium configuration $q_i = 0$. They may also describe approximately the motion of certain continuous linear systems whose governing partial differential equations may be reduced to a finite number of ordinary differential equations by some appropriate discretization technique such as the Rayleigh-Ritz, Galerkin, finite difference, or finite element procedures.

The stability of the equilibrium state $q = \dot{q} = 0$ for elastic systems, when the parametric excitation is a deterministic harmonic function of time, i.e $\eta_{ij} = \varepsilon \cos \omega t$, was investigated by Mettler (1968), and it is well known that instabilities occur when the excitation frequency ω is in the neighborhood of the values ω_0/p, where p is a positive integer and ω_0 depends on the form of the coupling coefficients k_{ij}. Instabilities of the first kind arise for $\omega_0 = 2\omega_i$ and correspond to parametric resonance of the subharmonic type in which only the particular mode q_i is excited into motion. Regions of instabilities of the second kind are found for

$$\omega_0 = \begin{cases} \omega_i + \omega_j & (i \neq j) \quad \text{if } k_{ij}k_{ji} > 0 \\ |\omega_i - \omega_j| & (i \neq j) \quad \text{if } k_{ij}k_{ji} < 0 \end{cases} \tag{4}$$

In the present analysis, the damping terms β_{ij}, the cosine and sine spectral densities of the multiplicative processes $\eta_{ij}(t)$, $\eta_{rs}(t)$, respectively, $S_{\eta_{ij}\eta_{rs}}(\omega)$ and $\Psi_{\eta_{ij}\eta_{rs}}(\omega)$, $i,j,r,s = 1,2\ldots,n$, are considered to be of the order of some small quantity ε, $0 < \varepsilon \ll 1$. The relaxation kernel $R(t)$ is assumed to be integrable such that $\int_0^\infty R(t)\,\mathrm{d}t < \infty$, $\int_0^\infty tR(t)\,\mathrm{d}t < \infty$, and the terms $\int_0^\infty e_{ii}R(t)\sin\omega_i\tau\,\mathrm{d}\tau$, $i = 1,2,\ldots,n$ are considered to be of order ε. The terms $\int_0^\infty R(t)\sin\omega_i\tau\,\mathrm{d}\tau$, $i = 1,2,\ldots,n$, are the sine transform functions of the relaxation kernel. Therefore, the stochastic averaging method may be used to replace the system of equations (1) by approximate Itô equations. The non-dimensional cosine and sine cross-spectral densities, $S_{\eta_{ij}\eta_{rs}}(\omega)$ and $\Psi_{\eta_{ij}\eta_{rs}}(\omega)$, respectively, are given by

$$\begin{aligned} S_{\eta_{ij}\eta_{rs}}(\omega) &= \sum_{\ell,m=1}^{N} c_{ij\ell}c_{rsm}S_{\xi_\ell\xi_m}(\omega) \\ \Psi_{\eta_{ij}\eta_{rs}}(\omega) &= \sum_{\ell,m=1}^{N} c_{ij\ell}c_{rsm}\Psi_{\xi_\ell\xi_m}(\omega) \end{aligned} \tag{5}$$

The non-dimensional functions $S_{\xi_\ell\xi_m}(\omega)$ and $\Psi_{\xi_\ell\xi_m}(\omega)$ are the cosine and sine cross-spectral densities of the ergodic stochastic processes $\xi_m(t)$ and $\xi_\ell(t)$, $\ell,m = 1,2,\ldots,N$, and are defined as

$$\begin{aligned} S_{\xi_\ell\xi_m}(\omega) &= 2\int_0^\infty E[\xi_\ell(t)\xi_m(t+\tau)]\cos\omega\tau\,\mathrm{d}\tau \\ \Psi_{\xi_\ell\xi_m}(\omega) &= 2\int_0^\infty E[\xi_\ell(t)\xi_m(t+\tau)]\sin\omega\tau\,\mathrm{d}\tau \end{aligned} \tag{6}$$

where $E[.]$ denotes the expectation operator, ω, and τ represent the non-dimensional frequency and separation time, respectively. Using the transformation

$$q_i = a_i\cos\Theta_i, \quad \dot{q}_i = -a_i\omega_i\sin\Theta_i, \quad \Theta_i = \omega_i t + \theta_i, \quad i = 1,2,\ldots,n \tag{7}$$

and applying the method of variation of parameters, equations of motion in terms of the amplitudes $a_i(t)$ and the phases $\theta_i(t)$ of the response processes can be obtained as follows:

$$\begin{aligned}
\dot{a}_i(t) &= -\frac{2\sin\Theta_i(t)}{\omega_i}\sum_{j=1}^{n}\beta_{ij}a_j\omega_j\sin\Theta_j(t) + \sin\Theta_i(t)\sum_{j=1}^{n}k_{ij}a_j\eta_{ij}(t)\cos\Theta_j(t) \\
&\quad - \omega_i a_i\sin\Theta_i(t)\sum_{j=1}^{n}\int_0^t e_{ij}R(t-\tau)\cos\Theta_j(\tau)\,\mathrm{d}\tau \\
\dot{\theta}_i(t) &= -\frac{2\cos\Theta_i(t)}{a_i\omega_i}\sum_{j=1}^{n}\beta_{ij}a_j\omega_j\sin\Theta_j(t) + \cos\Theta_i(t)\sum_{j=1}^{n}k_{ij}\frac{a_j}{a_i}\eta_{ij}(t)\cos\Theta_j(t) \\
&\quad - \omega_i\cos\Theta_i(t)\sum_{j=1}^{n}\int_0^t e_{ij}R(t-\tau)\cos\Theta_j(\tau)\,\mathrm{d}\tau
\end{aligned} \tag{8}$$

As ε decreases to zero, the solution of the system of equations (8) converges in the weak sense and up to first order in ε to a diffusive Markov process whose governing Itô equations are of the form

$$\begin{aligned}
\mathrm{d}a_i &= m_{a_i}\mathrm{d}t + \sum_{j=1}^{n}\sigma_{ij}\mathrm{d}W_{a_j} \\
\mathrm{d}\theta_i &= m_{\theta_i}\mathrm{d}t + \sum_{j=1}^{n}\mu_{ij}\mathrm{d}W_{\theta_j}
\end{aligned} \tag{9}$$

where W_{a_j} and W_{θ_j} are mutually independent unit Wiener processes. The drift coefficients m_{a_i} and m_{θ_i} and the diffusion coefficients σ_{ij}, μ_{ij} are obtained by using the averaging procedure of Stratonovich (1963) for the non-viscoelastic terms and of Larianov (1969) for the viscoelastic terms and are given by

$$m_{a_i} = \left[-\beta_{ii} - \frac{1}{2}\omega_i e_{ii}R_s(\omega_i) + \frac{3}{16}k_{ii}^2 S_{\eta_{ii}\eta_{ii}}(2\omega_i) + \frac{1}{8}\sum_{\substack{j=1\\ j\neq i}}^{n}k_{ij}k_{ji}S^{-}_{\eta_{ij}\eta_{ji}}\right]a_i + \frac{1}{16}\sum_{\substack{j=1\\ j\neq i}}^{n}k_{ij}^2 S^{+}_{\eta_{ij}\eta_{ij}}\frac{a_j^2}{a_i}$$

$$m_{\theta_i} = -\frac{1}{8}\sum_{\substack{j=1\\ j\neq i}}^{n}k_{ij}k_{ji}\Psi^{-}_{\eta_{ij}\eta_{ji}} - \frac{1}{8}k_{ii}^2\Psi_{\eta_{ii}\eta_{ii}}(2\omega_i) - \frac{1}{2}\omega_i e_{ii}R_c(\omega_i)$$

$$\begin{aligned}
[\sigma\sigma^T]_{ii} &= \frac{1}{8}k_{ii}^2 S_{\eta_{ii}\eta_{ii}}(2\omega_i)a_i^2 + \frac{1}{8}\sum_{\substack{j=1\\ j\neq i}}^{n}k_{ij}^2 S^{+}_{\eta_{ij}\eta_{ij}}a_j^2 \\
[\sigma\sigma^T]_{ij} &= \frac{1}{8}k_{ij}k_{ji}S^{-}_{\eta_{ij}\eta_{ji}}a_i a_j \quad i\neq j \\
[\mu\mu^T]_{ii} &= \frac{1}{8}k_{ii}^2[2S_{\eta_{ii}\eta_{ii}}(0) + S_{\eta_{ii}\eta_{ii}}(2\omega_i)] + \frac{1}{8}\sum_{\substack{j=1\\ j\neq i}}^{n}k_{ij}^2 S^{+}_{\eta_{ij}\eta_{ij}}\frac{a_j^2}{a_i^2} \\
[\mu\mu^T]_{ij} &= \frac{1}{4}k_{ii}k_{jj}S_{\eta_{ii}\eta_{jj}}(0) + \frac{1}{8}k_{ij}k_{ji}S^{+}_{\eta_{ij}\eta_{ji}} \quad i\neq j
\end{aligned} \tag{10}$$

The non-dimensional functions $S^{\pm}_{\eta_{ij}\eta_{ji}}$, $S^{\pm}_{\eta_{ij}\eta_{ij}}$, $\Psi\pm_{\eta_{ij}\eta_{ji}}$ $i \neq j$ are defined as

$$\begin{aligned}
S^{\pm}_{\eta_{ij}\eta_{ji}} &= \sum_{\ell,m=1}^{N} c_{ij\ell}c_{jim}[S_{\xi_\ell\xi_m}(\omega_i+\omega_j) \pm S_{\xi_\ell\xi_m}(\omega_i-\omega_j)] \\
S^{\pm}_{\eta_{ij}\eta_{ij}} &= \sum_{\ell,m=1}^{N} c_{ij\ell}c_{ijm}[S_{\xi_\ell\xi_m}(\omega_i+\omega_j) \pm S_{\xi_\ell\xi_m}(\omega_i-\omega_j)] \\
\Psi^{\pm}_{\eta_{ij}\eta_{ji}} &= \sum_{\ell,m=1}^{N} c_{ij\ell}c_{jim}[\Psi_{\xi_\ell\xi_m}(\omega_i+\omega_j) \pm \Psi_{\xi_\ell\xi_m}(\omega_i-\omega_j)]
\end{aligned} \tag{11}$$

The non-dimensional one sided Fourier sine and cosine transforms of the relaxation kernel are given by

$$\begin{aligned}
R_s(\omega_i) &= \int_0^\infty R(\tau)\sin\omega_i\tau\,\mathrm{d}\tau \\
R_c(\omega_i) &= \int_0^\infty R(\tau)\cos\omega_i\tau\,\mathrm{d}\tau
\end{aligned} \tag{12}$$

Since it is difficult to study the n-degrees of freedom system of Eq. (1) in its general form, the analysis from now on will be restricted to a two-degrees of freedom system. The results obtained for the two-degrees of freedom system may be generalized to n-degrees of freedom systems under certain conditions on the spectral density distribution of the ergodic stochastic processes $\xi_m(t)$. Without loss of generality, it is always possible to choose a suitable coordinate scaling such that $k_{12} = \pm k_{21} = k > 0$, and with the product $|k_{12}k_{21}|$ invariant under the scaling. For the two-degrees of freedom system, the amplitude equations corresponding to Eq. (9) become

$$\begin{aligned}
\mathrm{d}a_1 &= m_{a_1}\mathrm{d}t + \sigma_{11}\,\mathrm{d}W_{a_1} + \sigma_{12}\,\mathrm{d}W_{a_2} \\
\mathrm{d}a_2 &= m_{a_2}\mathrm{d}t + \sigma_{21}\,\mathrm{d}W_{a_1} + \sigma_{22}\,\mathrm{d}W_{a_2}
\end{aligned} \tag{13}$$

The averaged amplitude vector (a_1, a_2) is a two-dimensional diffusion process and it can easily be shown that the coefficients of the right side of Eq. (13) are homogeneous in a_1, a_2 of degree one. The procedure of Khasminskii (1967) can therefore be employed to obtain the largest Lyapunov exponent of the amplitude process. By using the logarithmic polar transformation

$$\rho = \log(a_1^2 + a_2^2)^{1/2}, \quad \phi = \tan^{-1}(a_2/a_1), \quad \phi\,\epsilon\,(0, \pi/2) \tag{14}$$

and employing Itô's differential rule, the following Itô equations for the functions ρ and ϕ can be obtained as follows:

$$\begin{aligned}
\mathrm{d}\rho &= Q(\phi)\mathrm{d}t + \sum_{j=1}^{2}\left(\sigma_{a_{1j}}\frac{\partial\rho}{\partial a_1} + \sigma_{a_{2j}}\frac{\partial\rho}{\partial a_2}\right)\mathrm{d}W_j(t) \\
\mathrm{d}\phi &= \Phi(\phi)\mathrm{d}t + \sum_{j=1}^{2}\left(\sigma_{a_{1j}}\frac{\partial\phi}{\partial a_1} + \sigma_{a_{2j}}\frac{\partial\phi}{\partial a_2}\right)\mathrm{d}W_j(t)
\end{aligned} \tag{15}$$

where the drift coefficients $Q(\phi)$ and $\Phi(\phi)$ in Eq. (15) are given by

$$\begin{aligned}
Q(\phi) &= \lambda_1\cos^2\phi + \lambda_2\sin^2\phi \pm \frac{1}{8}k^2 S^{-}_{\eta_{12}\eta_{21}} + \Sigma_{\phi\phi}(\phi) \\
\Phi(\phi) &= -\frac{1}{2}(\lambda_1 - \lambda_2)\sin 2\phi + \frac{1}{2}b\sin 4\phi + \frac{1}{16}k^2\left(S^{+}_{\eta_{12}\eta_{12}} + S^{+}_{\eta_{21}\eta_{21}}\right)\cot 2\phi \\
&\quad + f\cos 2\phi\cot 2\phi
\end{aligned} \tag{16}$$

while the square of the diffusion coefficient for the $\phi(t)$-process is given by

$$\Sigma_{\phi\phi}(\phi) = a + f\cos 2\phi - b\cos^2 2\phi \tag{17}$$

where the constants $\lambda_i = -\beta_{ii} - \frac{1}{2}\omega_i e_{ii} R_s(\omega_i) + \frac{1}{8}k_{ii}^2 S_{\eta_{ii}\eta_{ii}}(2\omega_i)$, and a, b and f are found to be

$$\begin{aligned} a &= \frac{1}{32}[k_{11}^2 S_{\eta_{11}\eta_{11}}(2\omega_1) + k_{22}^2 S_{\eta_{22}\eta_{22}}(2\omega_2) + (S^+_{\eta_{12}\eta_{12}} + S^+_{\eta_{21}\eta_{21}} \mp 2S^-_{\eta_{12}\eta_{21}})k^2] \\ b &= \frac{1}{32}[k_{11}^2 S_{\eta_{11}\eta_{11}}(2\omega_1) + k_{22}^2 S_{\eta_{22}\eta_{22}}(2\omega_2) - (S^+_{\eta_{12}\eta_{12}} + S^+_{\eta_{21}\eta_{21}} \pm 2S^-_{\eta_{12}\eta_{21}})k^2] \\ f &= \frac{1}{16}k^2(S^+_{\eta_{12}\eta_{12}} - S^+_{\eta_{21}\eta_{21}}) \end{aligned} \tag{18}$$

In the expressions for the constants a and b, the upper sign (+) is taken when $k_{12} = k_{21} = k$ and the lower sign(-) is taken when $k_{12} = -k_{21} = k$.

For a non-singular diffusion, i.e. when the diffusion coefficient in the Itô equation of the ϕ process is not equal to zero, the probability density, $\mu(\phi)$, of the invariant measure of the ϕ-process is governed by the following Fokker–Planck equation:

$$\frac{d}{d\phi}\left\{\frac{1}{2}\frac{d}{d\phi}[\Sigma_{\phi\phi}(\phi)\mu(\phi)] - \Phi(\phi)\mu(\phi)\right\} = 0 \tag{19}$$

By solving Eq. (19), one obtains

$$\mu(\phi) = \frac{C}{\Sigma_{\phi\phi}(\phi)U(\phi)} - \frac{G_0}{\Sigma_{\phi\phi}(\phi)U(\phi)}\int U(\phi)\,d\phi \tag{20}$$

where C and G_0 are the constants of integration and $U(\phi)$ is given by

$$U(\phi) = \exp\left[-2\int \frac{\Phi(\phi)}{\Sigma_{\phi\phi}(\phi)}\,d\phi\right] \tag{21}$$

Upon substituting for $\Phi(\phi)$ and $\Sigma_{\phi\phi}(\phi)$ from Eqs. (16) and (17), the following expression for $U(\phi)$ can be obtained

$$U(\phi) = \frac{1}{\sin 2\phi}\exp\left\{\int \frac{(\lambda_1 - \lambda_2)\sin 2\phi\,d\phi}{a + f\cos 2\phi - b\cos^2 2\phi}\right\} \tag{22}$$

The objective of the present study is to investigate the stochastic stability of a non-gyroscopic viscoelastic system. One of the most important parameters in studying stochastic stability of dynamic systems is the largest Lyapunov exponent, which characterizes the rate of exponential growth of the system response with the passage of time. If the maximum exponent is positive, the system is unstable with probability one and if it is negative, the system is stable with probability one. The vanishing of the expression for the top Lyapunov exponent yields the almost-sure stability boundaries in the system parameter space.

3. Calculation of Lyapunov exponent

Case 1: $\eta_{11}(t) \neq \eta_{12}(t) \neq \eta_{21}(t) \neq \eta_{22}(t)$. The integral in Eq. (22) depends on the value of $\Delta = -4b/a - (f/a)^2$. If there is no accumulation of probability mass at the boundaries, the ϕ-process is ergodic throughout the interval $0 < \phi < \pi/2$, and the probability density, $\mu(\phi)$, of the invariant measure can be obtained as:

For $\Delta < 0$,

$$\mu(\phi) = \frac{C \sin 2\phi}{\Sigma_{\phi\phi}(\phi)} \exp\left(\frac{\lambda_2 - \lambda_1}{a\sqrt{-\Delta}} \tanh^{-1} \frac{f - 2b\cos 2\phi}{a\sqrt{-\Delta}}\right)$$

For $\Delta > 0$,

$$\mu(\phi) = \frac{C \sin 2\phi}{\Sigma_{\phi\phi}(\phi)} \exp\left(-\frac{\lambda_2 - \lambda_1}{a\sqrt{\Delta}} \tan^{-1} \frac{f - 2b\cos 2\phi}{a\sqrt{\Delta}}\right)$$

For $\Delta = 0$,

$$\mu(\phi) = \frac{C \sin 2\phi}{\Sigma_{\phi\phi}(\phi)} \exp\left(\frac{\lambda_2 - \lambda_1}{f - 2b\cos 2\phi}\right) \tag{23}$$

where the constant C is determined as follows from the normalizing condition:

For $\Delta < 0$,

$$C = \frac{(\lambda_2 - \lambda_1)}{\left[\exp\left(\frac{\lambda_2-\lambda_1}{a\sqrt{-\Delta}} \tanh^{-1}\gamma_2\right) - \exp\left(\frac{\lambda_2-\lambda_1}{a\sqrt{-\Delta}} \tanh^{-1}\gamma_1\right)\right]}$$

For $\Delta > 0$,

$$C = \frac{(\lambda_2 - \lambda_1)}{\left[\exp\left(-\frac{\lambda_2-\lambda_1}{a\sqrt{\Delta}} \tan^{-1}\gamma_2\right) - \exp\left(-\frac{\lambda_2-\lambda_1}{a\sqrt{\Delta}} \tan^{-1}\gamma_1\right)\right]}$$

For $\Delta = 0$,

$$C = \frac{(\lambda_2 - \lambda_1)}{\left[\exp\left(\frac{\lambda_2-\lambda_1}{(f+2b)}\right) - \exp\left(\frac{\lambda_2-\lambda_1}{(f-2b)}\right)\right]} \tag{24}$$

where the constants γ_1, γ_2 are defined by

$$\gamma_1 = \frac{f - 2b}{a\sqrt{|\Delta|}}, \qquad \gamma_2 = \frac{f + 2b}{a\sqrt{|\Delta|}} \tag{25}$$

Employing Khasminskii's (1967) technique, the largest Lyapunov exponent is given, with probability one, by

$$\lambda = E[Q(\phi)] = \int_0^{\pi/2} Q(\phi)\mu(\phi)\,\mathrm{d}\phi \tag{26}$$

After performing the indicated integration, the following expressions for the largest Lyapunov exponent can be obtained:

For $\Delta < 0$,

$$\lambda = \frac{1}{2}(\lambda_2 - \lambda_1) \frac{\left[\exp\left(\frac{\lambda_2-\lambda_1}{a\sqrt{-\Delta}} \tanh^{-1}\gamma_2\right) + \exp\left(\frac{\lambda_2-\lambda_1}{a\sqrt{-\Delta}} \tanh^{-1}\gamma_1\right)\right]}{\left[\exp\left(\frac{\lambda_2-\lambda_1}{a\sqrt{-\Delta}} \tanh^{-1}\gamma_2\right) - \exp\left(\frac{\lambda_2-\lambda_1}{a\sqrt{-\Delta}} \tanh^{-1}\gamma_1\right)\right]} + \frac{1}{2}(\lambda_1 + \lambda_2) \pm \frac{1}{8}k^2 S^-_{\eta_{12}\eta_{21}}$$

For $\varDelta > 0$,

$$\lambda = \frac{1}{2}(\lambda_2 - \lambda_1)\frac{\left[\exp\left(-\frac{\lambda_2-\lambda_1}{a\sqrt{\varDelta}}\tan^{-1}\gamma_2\right)+\exp\left(-\frac{\lambda_2-\lambda_1}{a\sqrt{\varDelta}}\tan^{-1}\gamma_1\right)\right]}{\left[\exp\left(-\frac{\lambda_2-\lambda_1}{a\sqrt{\varDelta}}\tan^{-1}\gamma_2\right)-\exp\left(-\frac{\lambda_2-\lambda_1}{a\sqrt{\varDelta}}\tan^{-1}\gamma_1\right)\right]} + \frac{1}{2}(\lambda_1+\lambda_2) \pm \frac{1}{8}k^2 S^-_{\eta_{12}\eta_{21}}$$

For $\varDelta = 0$,

$$\lambda = \frac{1}{2}(\lambda_2 - \lambda_1)\frac{\left[\exp\left(\frac{\lambda_2-\lambda_1}{(f+2b)}\right)+\exp\left(\frac{\lambda_2-\lambda_1}{(f-2b)}\right)\right]}{\left[\exp\left(\frac{\lambda_2-\lambda_1}{(f+2b)}\right)-\exp\left(\frac{\lambda_2-\lambda_1}{(f-2b)}\right)\right]} + \frac{1}{2}(\lambda_1+\lambda_2) \pm \frac{1}{8}k^2 S^-_{\eta_{12}\eta_{21}} \tag{27}$$

Case 2: $\eta_{ij}(t) = \eta(t)$, $i,j = 1,2$. For this case, the constant $f = 0$, while the constants a and b are given in terms of the spectral densities of the ergodic stochastic processes $\xi_m(t)$ and $\xi_\ell(t)$ as the following:

$$\begin{aligned} a &= \frac{1}{32}\sum_{\ell,m=1}^{N} c_{11\ell}c_{11m}[k_{11}^2 S_{\xi_\ell\xi_m}(2\omega_1) + k_{22}^2 S_{\xi_\ell\xi_m}(2\omega_2) + 4k^2 S_{\xi_\ell\xi_m}(\omega_1 \mp \omega_2)] \\ b &= \frac{1}{32}\sum_{\ell,m=1}^{N} c_{11\ell}c_{11m}[k_{11}^2 S_{\xi_\ell\xi_m}(2\omega_1) + k_{22}^2 S_{\xi_\ell\xi_m}(2\omega_2) - 4k^2 S_{\xi_\ell\xi_m}(\omega_1 \pm \omega_2)] \end{aligned} \tag{28}$$

Utilizing the same procedure as for Case (1), the probability density, $\mu(\phi)$, of the invariant measure and the largest Lyapunov exponent λ are found to be:

For $b > 0$,

$$\mu(\phi) = \frac{C\sin 2\phi}{\Sigma_{\phi\phi}(\phi)}\exp\left(\frac{\lambda_1-\lambda_2}{\sqrt{\varDelta_0}}\cosh^{-1}\frac{a+b\cos^2 2\phi}{a-b\cos^2 2\phi}\right)$$

where the constant $\varDelta_0 = 16ab$ and the normalization constant C is obtained as:

$$C = \frac{(\lambda_1-\lambda_2)}{2\sinh\left(\frac{\lambda_1-\lambda_2}{\sqrt{\varDelta_0}}\cosh^{-1}\frac{a+b}{a-b}\right)}$$

$$\lambda = \frac{1}{2}(\lambda_1-\lambda_2)\coth\left(\frac{\lambda_1-\lambda_2}{\sqrt{\varDelta_0}}\cosh^{-1}\frac{a+b}{a-b}\right) + \frac{1}{2}(\lambda_1+\lambda_2) \pm \frac{1}{8}k^2\sum_{\ell,m=1}^{N} c_{11\ell}c_{11m}S^-_{\xi_\ell\xi_m}$$

For $b < 0$,

$$\mu(\phi) = \frac{C\sin 2\phi}{\Sigma_{\phi\phi}(\phi)}\exp\left(-\frac{\lambda_1-\lambda_2}{\sqrt{-\varDelta_0}}\cos^{-1}\frac{a+b\cos^2 2\phi}{a-b\cos^2 2\phi}\right)$$

$$C = \frac{(\lambda_1-\lambda_2)}{2\sinh\left(\frac{\lambda_1-\lambda_2}{\sqrt{-\varDelta_0}}\cos^{-1}\frac{a+b}{a-b}\right)}$$

$$\lambda = \frac{1}{2}(\lambda_1-\lambda_2)\coth\left(\frac{\lambda_1-\lambda_2}{\sqrt{-\varDelta_0}}\cos^{-1}\frac{a+b}{a-b}\right) + \frac{1}{2}(\lambda_1+\lambda_2) \pm \frac{1}{8}k^2\sum_{\ell,m=1}^{N} c_{11\ell}c_{11m}S^-_{\xi_\ell\xi_m}$$

For $b = 0$,

$$\begin{aligned}
\mu(\phi) &= \frac{C\sin 2\phi}{\Sigma_{\phi\phi}(\phi)} \exp\left(\frac{\lambda_1 - \lambda_2}{2a}\cos 2\phi\right) \\
C &= \frac{(\lambda_1 - \lambda_2)}{\sinh\left(\frac{\lambda_1-\lambda_2}{2a}\right)} \\
\lambda &= \frac{1}{2}(\lambda_1 - \lambda_2)\coth\left(\frac{4(\lambda_1 - \lambda_2)}{k^2\sum_{\ell,m=1}^{N} c_{11\ell}c_{11m}S^{+}_{\xi_\ell\xi_m}}\right) + \frac{1}{2}(\lambda_1 + \lambda_2) \\
&\quad \pm \frac{1}{8}k^2\sum_{\ell,m=1}^{N} c_{11\ell}c_{11m}S^{-}_{\xi_\ell\xi_m}
\end{aligned} \tag{29}$$

The expressions of Eq. (29) are in the same form as those obtained earlier by Ariaratnam and Abdelrahman (2001), but here are obtained as a special case when the multiple random excitation $\eta_{ij}(t)$, $i, j = 1, 2$, has a specific form.

4. Singular case

The point $\phi = \phi_0$ is considered to be a singular point when the square of the diffusion coefficient $\Sigma_{\phi\phi}(\phi)$ of the phase process ϕ vanishes at that point. For the case $\eta_{11}(t) \neq \eta_{12}(t) \neq \eta_{21}(t) \neq \eta_{22}(t)$, the point $\phi = \pi/4$ is a singular point when the following conditions are satisfied

$$k_{11} = k_{22} = 0, \quad S_{\xi_\ell\xi_m}(\omega_1 \pm \omega_2) = 0, \quad c_{12\ell} = -c_{21\ell}, \quad c_{12m} = -c_{21m} \tag{30}$$

where $\ell, m = 1, 2, \ldots, N$, the upper sign (+) is taken when $k_{12} = k_{21} = k$, and the lower sign(-) when $k_{12} = -k_{21} = k$. Under the conditions of Eq. (30), the drift coefficients $Q(\phi)$, $\Phi(\phi)$ and the diffusion coefficient $\Sigma_{\phi\phi}(\phi)$ are given by

$$\begin{aligned}
Q(\phi) &= -\left(\beta_{11} + \frac{1}{2}\omega_1 e_{11}R_s(\omega_1)\right)\cos^2\phi - \left(\beta_{22} + \frac{1}{2}\omega_2 e_{22}R_s(\omega_2)\right)\sin^2\phi \\
&\quad + \frac{1}{8}k^2\sum_{\ell,m=1}^{N} c_{12\ell}c_{12m}S_{\xi_\ell\xi_m}(\omega_1 \mp \omega_2) + \Sigma_{\phi\phi}(\phi) \\
\Phi(\phi) &= \frac{1}{4}[2(\beta_{11} - \beta_{22}) + \omega_1 e_{11}R_s(\omega_1) - \omega_2 e_{22}R_s(\omega_2)]\sin 2\phi \\
&\quad + \frac{1}{8}k^2\sum_{\ell,m=1}^{N} c_{12\ell}c_{12m}S_{\xi_\ell\xi_m}(\omega_1 \mp \omega_2)\cot 2\phi\cos^2 2\phi \\
\Sigma_{\phi\phi}(\phi) &= \frac{1}{8}k^2\sum_{\ell,m=1}^{N} c_{12\ell}c_{12m}S_{\xi_\ell\xi_m}(\omega_1 \mp \omega_2)\cos^2 2\phi
\end{aligned} \tag{31}$$

The nature of the singular point $\phi = \phi_0$ is determined by the sign of the drift coefficient $\Phi(\phi)$ at that point (Mitchell and Kozin, 1974). Applying this criterion to the singular point $\phi_0 = \pi/4$, the following three cases are considered.

(i) If $2\beta_{11} + \omega_1 e_{11}R_s(\omega_1) > 2\beta_{22} + \omega_2 e_{22}R_s(\omega_2)$, the singular point $\phi_0 = \pi/4$ is a right or forward shunt. This implies that even if an initial point ϕ_0 is in the left-half interval $(0, \pi/4)$, it will eventually be shunted across to the right-half interval $(\pi/4, \pi/2)$ and remain there forever. The probability density $\mu(\phi)$ of the invariant measure is confined to the right-half of the interval $0 < \phi < \pi/2$ and is governed by the Fokker–Planck equation whose solution is now of the form:

$$\mu(\phi) = \begin{cases} 0 & 0 < \phi < \pi/4 \\ \frac{C}{\Sigma_{\phi\phi}(\phi)U(\phi)} & \pi/4 < \phi < \pi/2 \end{cases} \tag{32}$$

where C is the normalizing constant. The largest Lyapunov exponent is obtained as

$$\lambda = -\frac{1}{2}[2\beta_{22} + \omega_2 e_{22} R_s(\omega_2)] + \frac{1}{8}k^2 \sum_{\ell,m=1}^{N} c_{12\ell} c_{12m} S_{\xi_\ell \xi_m}(\omega_1 \mp \omega_2) \tag{33}$$

(ii) If $2\beta_{11} + \omega_1 e_{11} R_s(\omega_1) < 2\beta_{22} + \omega_2 e_{22} R_s(\omega_2)$, the singular point $\phi_0 = \pi/4$ is a left or backward shunt. This implies that even if an initial point ϕ_0 is in the right-half interval $(\pi/4, \pi/2)$, it will eventually be shunted across to the left-half interval $(0, \pi/4)$ and remain there forever. The probability density $\mu(\phi)$ of the invariant measure is concentrated in the left-half of the interval $0 < \phi < \pi/2$ and is governed by the Fokker–Planck equation whose solution is now of the form:

$$\mu(\phi) = \begin{cases} \frac{C}{\Sigma_{\phi\phi}(\phi)U(\phi)} & 0 < \phi < \pi/4 \\ 0 & \pi/4 < \phi < \pi/2 \end{cases} \tag{34}$$

The largest Lyapunov exponent is now given by

$$\lambda = -\frac{1}{2}[2\beta_{11} + \omega_1 e_{11} R_s(\omega_1)] + \frac{1}{8}k^2 \sum_{\ell,m=1}^{N} c_{12\ell} c_{12m} S_{\xi_\ell \xi_m}(\omega_1 \mp \omega_2) \tag{35}$$

(iii) If $2\beta_{11} + \omega_1 e_{11} R_s(\omega_1) = 2\beta_{22} + \omega_2 e_{22} R_s(\omega_2)$, the singular point $\phi_0 = \pi/4$ is a trap. This implies that regardless of where the initial point ϕ_0 is situated, it will eventually be attracted to the point $\phi_0 = \pi/4$ and remain there forever. In this case, the probability density, $\mu(\phi)$, of the invariant measure is the Dirac delta function concentrated at $\phi = \pi/4$, i.e $\mu = \delta(\phi - \pi/4)$, and the largest Lyapunov exponent is.

$$\lambda = -\frac{1}{2}[2\beta_{11} + \omega_1 e_{11} R_s(\omega_1)] + \frac{1}{8}k^2 \sum_{\ell,m=1}^{N} c_{12\ell} c_{12m} S_{\xi_r \xi_r}(\omega_1 \mp \omega_2) \tag{36}$$

For the case $\eta_{ij} = \eta$, $i, j = 1, 2$, the point $\phi = \pi/4$ is a singular point when the following conditions are satisfied

$$k_{11} = k_{22} = 0, \quad S_{\xi_\ell \xi_m}(\omega_1 \mp \omega_2) = 0, \quad c_{12\ell} = c_{21\ell}, \quad c_{12m} = c_{21m} \tag{37}$$

where $\ell, m = 1, 2, \ldots, r$, the upper sign (-) is taken when $k_{12} = k_{21} = k$, and the lower sign(+) when $k_{12} = -k_{21} = k$. Similarly, it can be shown that the largest Lyapunov exponents are of the same form as those for the case $\eta_{11} \neq \eta_{12} \neq \eta_{21} \neq \eta_{22}$ but with the term $S_{\xi_\ell \xi_m}(\omega_1 \mp \omega_2)$ replaced by $S_{\xi_\ell \xi_m}(\omega_1 \pm \omega_2)$.

The results obtained in the previous sections are applied to investigate the stability of a deep rectangular viscoelastic beam acted upon by stochastically varying concentrated load and end moments. Both cases of non-follower and follower concentrated load are considered.

5. Flexural–torsional stability of a viscoelastic beam

5.1. Equation of motion

Consider a beam in the form of a rectangular strip supported at its ends, with principal flexural rigidities EI_x, EI_y and torsional rigidity GJ. The beam is of length L, width w, thickness h such that $w/h \ll 1$, mass per

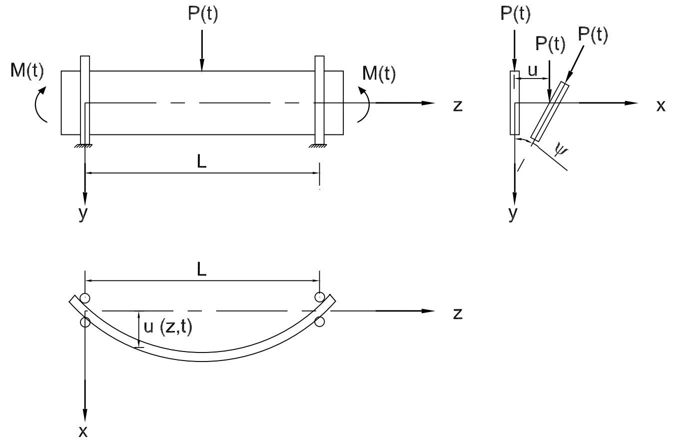

Fig. 1. Beam under central force and end moments.

unit length m, translational and rotational viscous damping coefficients per unit length D_u, D_ψ, respectively, and with a polar radius of gyration of the cross-section r.

The beam is bent in its plane by a stochastically varying static central load $P = P_s + P(t_1)$ and end moments $M = M_s + M(t_1)$. It is assumed that during deformation the ends can rotate freely with respect to a plane parallel to the (x, y)-plane, but are prevented from rotation with respect to the z-axis by some constraint (see Fig. 1); thus, it is assumed that the lateral deflection is accompanied by a beam twist. Let the deflection of the beam in the x-direction be denoted by u and the angle of twist through which the cross-section rotates by ψ. The deflection in the y-direction is being neglected, since it is assumed that $EI_x \gg EI_y$. The angle of twist is considered positive when the rotation is in the direction from the x-axis to the y-axis axis.

5.2. Non-follower force case

For a thin elastic beam under central force and end moments, the flexural and torsional equations of motion, neglecting warping rigidity, (see e.g. Timoshenko, 1936; Bolotin, 1964; Fu and Nemat-Nasser, 1972; Ariaratnam et al., 1992) are given as follows:

$$\begin{aligned} &EI_y \frac{\partial^4 u}{\partial z^4} + \frac{\partial^2 (M_x \psi)}{\partial z^2} + m \frac{\partial^2 u}{\partial t_1^2} + D_u \frac{\partial u}{\partial t_1} = 0 \\ &- GJ \frac{\partial^2 \psi}{\partial z^2} + M_x \frac{\partial^2 u}{\partial z^2} + mr^2 \frac{\partial^2 \psi}{\partial t_1^2} + D_\psi \frac{\partial \psi}{\partial t_1} = 0 \end{aligned} \tag{38}$$

where

$$M_x = \begin{cases} \frac{1}{2} Pz + M & 0 \leqslant z \leqslant L/2 \\ \frac{1}{2} P(L - z) + M & L/2 \leqslant z \leqslant L \end{cases} \tag{39}$$

with boundary conditions

$$\begin{aligned} &u(0, t_1) = u(L, t_1) = \left. \frac{\partial^2 u}{\partial z^2} \right|_{z=0} = \left. \frac{\partial^2 u}{\partial z^2} \right|_{z=L} = 0 \\ &\psi(0, t_1) = \psi(L, t_1) = 0 \end{aligned} \tag{40}$$

According to the Boltzmann superposition principle, a constitutive relation for a linear viscoelastic material under three dimensional loading can be obtained by replacing the bulk modulus κ and the shear modulus G by appropriate Voltera operators (Drozdov, 1996). If in addition the material demonstrates elastic bulk response, the bulk modulus is time independent and the following constitutive relation for a linear viscoelastic material can be used:

$$\begin{aligned} &\sigma = 3\kappa\epsilon, \quad \hat{s} = 2G(1-\mathbf{R})\hat{e} \\ &\hat{\sigma} = \sigma + \hat{s}, \quad \hat{\epsilon} = \epsilon + \hat{e} \end{aligned} \tag{41}$$

where σ, ϵ, and $\hat{s}$, $\hat{e}$ are the first invariants and the deviatoric parts of the stress tensor $\hat{\sigma}$ and the strain tensor $\hat{\epsilon}$, respectively, and $\mathbf{R}$ is the relaxation operator given by Eq. (2). From the first of Eq. (41) and by employing the correspondence principle, the elastic moduli E and G can be replaced by the Voltera operators $E(1-\mathbf{R})$ and $G(1-\mathbf{R})$, respectively. Therefore, for a viscoelastic beam the following equations of motion neglecting warping rigidity can be obtained:

$$\begin{aligned} &EI_y(1-\mathbf{R})\frac{\partial^4 u}{\partial z^4} + \frac{\partial^2(M_x\psi)}{\partial z^2} + m\frac{\partial^2 u}{\partial t_1^2} + D_u\frac{\partial u}{\partial t_1} = 0 \\ &-GJ(1-\mathbf{R})\frac{\partial^2\psi}{\partial z^2} + M_x\frac{\partial^2 u}{\partial z^2} + mr^2\frac{\partial^2\psi}{\partial t_1^2} + D_\psi\frac{\partial\psi}{\partial t_1} = 0 \end{aligned} \tag{42}$$

with boundary conditions

$$\begin{aligned} &u(0,t_1) = u(L,t_1) = (\partial^2 u/\partial z^2)|_{z=0} = (\partial^2 u/\partial z^2)|_{z=L} = 0 \\ &\psi(0,t_1) = \psi(L,t_1) = 0 \end{aligned} \tag{43}$$

Eq. (42) are difficult to solve in this form and therefore an approximate solution can be sought by using some discretizing technique such as the Galerkin method. If only the fundamental modes are considered, the above boundary conditions are satisfied by taking

$$\begin{aligned} &u(z,t_1) = c^* r q_1(t_1)\sin(\pi z/L) \\ &\psi(z,t_1) = q_2(t_1)\sin(\pi z/L) \end{aligned} \tag{44}$$

where c^* is a parameter to be chosen to facilitate a suitable coordinate scaling. Substituting for u and ψ from Eq. (44) into the equations of motion (42), employing the Galerkin method and introducing the non-dimensional time scale $t = \nu t_1$, where the frequency ν is given by $\nu = \frac{\pi^2}{L^2}\sqrt{\frac{EI_y}{m}}$, the following non-dimensional system of equations can be obtained.

$$\begin{aligned} &\ddot{q}_1 + \frac{L^2 D_u}{\pi^2\sqrt{EI_y m}}\dot{q}_1 + (1-\mathbf{R})q_1 - \frac{\gamma_{01}}{c^*}q_2 - \frac{1}{c^*}(\xi_1+\xi_2)q_2 = 0 \\ &\ddot{q}_2 + \frac{L^2 D_\psi}{\pi^2 r^2\sqrt{EI_y m}}\dot{q}_2 + \frac{GJL^2}{\pi^2 r^2 EI_y}(1-\mathbf{R})q_2 - c^*\gamma_{01}q_1 - c^*(\xi_1+\xi_2)q_1 = 0 \end{aligned} \tag{45}$$

where $\xi_1 = \frac{(4+\pi^2)L^3 P(t)}{8\pi^4 r EI_y}$ and $\xi_2 = \frac{L^2 M(t)}{\pi^2 r EI_y}$. Upon using the transformation

$$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \to \begin{bmatrix} 1 & \gamma_{01}/(c^*(\omega_1^2-\omega_2^2)) \\ c^*(1-\omega_1^2)/\gamma_{01} & (1-\omega_2^2)/(\omega_1^2-\omega_2^2) \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \tag{46}$$

where

$$\begin{aligned}\omega_{1,2}^2 &= \frac{1}{2}\left(1 + L^2GJ/(\pi^2r^2EI_y) \mp \sqrt{\left(1 - L^2GJ/(\pi^2r^2EI_y)\right)^2 + 4\gamma_{01}^2}\right)\\ \gamma_{01} &= L^2((4+\pi^2)P_sL + 8\pi^2M_s)/(8r\pi^4EI_y)\end{aligned} \tag{47}$$

and pre-multiplying by the inverse of the transformation matrix of Eq. (46), non-dimensional equations of motion of the same form as Eq. (1) can be obtained. For this case, $k_{12} = k_{21} = k$ and the terms k_{11}, k_{22}, k and c^* are obtained as follows:

$$\begin{aligned}&k_{11} = 2\gamma_{01}/(\omega_1(\omega_1^2 - \omega_2^2)), \quad k_{22} = -2\gamma_{01}/(\omega_2(\omega_1^2 - \omega_2^2))\\ &|k_{12}k_{21}| = k^2 = \frac{|(\omega_1^2 - \omega_2^2)^2 - 4\gamma_{01}^2|}{\omega_1\omega_2(\omega_1^2 - \omega_2^2)^2}\\ &c^* = \frac{\gamma_{01}}{(\omega_1^2 - \omega_2^2)}\left|\frac{\omega_2(\gamma_{01}^2 - (1-\omega_2^2)^2)}{\omega_1(\gamma_{01}^2 - (1-\omega_1^2)^2)}\right|^{1/2}\end{aligned} \tag{48}$$

The multiplicative processes η_{ij} are such that $\eta_{ij}(t) = \eta(t) = \xi_1(t) + \xi_2(t),\ i,j = 1,2$. The corresponding damping and viscoelastic terms β_{ii} and e_{ii}, $i = 1,2$, of Eq. (1) are obtained as follows:

$$\begin{aligned}\beta_{11} &= (L^2/(2\pi^2\sqrt{EI_ym}(\omega_1^2 - \omega_2^2)))((\omega_1^2 - 1)D_\psi/r^2 - (\omega_2^2 - 1)D_u)\\ \beta_{22} &= (L^2/(2\pi^2\sqrt{EI_ym}(\omega_1^2 - \omega_2^2)))((\omega_1^2 - 1)D_u - (\omega_2^2 - 1)D_\psi/r^2)\\ e_{11} &= ((\omega_1^2 - 1)(\omega_1^2 + \omega_2^2 - 1) - \omega_2^2 + 1)/(\omega_1^2(\omega_1^2 - \omega_2^2))\\ e_{22} &= (\omega_1^2 - 1 - (\omega_2^2 - 1)(\omega_1^2 + \omega_2^2 - 1))/(\omega_2^2(\omega_1^2 - \omega_2^2))\end{aligned} \tag{49}$$

It is worth noting that the transformation matrix of Eq. (46), which is needed to diagonalize the original coupled system, is constructed from the eigenvectors of the stiffness matrix of the elastic system of equations (45). The frequencies $\omega_{1,2}$ are real provided $\frac{L^2GJ}{\pi^2r^2EI_y} > \gamma_{01}^2$; therefore, the static loads at which static buckling will occur are such that $\frac{P_s}{P_{\mathrm{cr}}} + \frac{M_s}{M_{\mathrm{cr}}} = 1$, where

$$P_{\mathrm{cr}} = \frac{8\pi^3}{(4+\pi^2)L^2}\sqrt{EI_yGJ}, \quad M_{\mathrm{cr}} = \frac{\pi}{L}\sqrt{EI_yGJ} \tag{50}$$

The constant coefficients which measure the contribution of the ergodic processes $\xi_m(t)$, $m = 1,2$, to the multiplicative processes $\eta_{ij}(t)$, $ij = 1,2$, are obtained as $c_{111} = c_{121} = c_{211} = c_{221} = 1$ and $c_{112} = c_{122} = c_{212} = c_{222} = 1$. The cross-damping and viscoelastic terms β_{ij} and e_{ij}, $i \neq j$, have no effect on the solution in the first approximation. Defining the non-dimensional parameter

$$\bar{\beta}_i = 2\left(2\beta_i^* - S_i - S_3\right)\eta_0, \quad i = 1,2 \tag{51}$$

where $\beta_i^* = \frac{2}{k^2S_{\eta\eta}^+}(2\beta_{ii} + \omega_ie_{ii}R_s(\omega_i))$ and

$$\eta_0 = \begin{cases}\frac{1}{(4-S_0^2)^{1/2}}\cos^{-1}(S_0/2) & b < 0\\ \frac{1}{(S_0^2-4)^{1/2}}\cosh^{-1}(S_0/2) & b > 0\\ \frac{1}{2} & b = 0\end{cases} \tag{52}$$

$$\begin{aligned}&S_0 = S_1 + S_2 - 2S_3, \quad S_1 = k_{11}^2S_{\eta\eta}(2\omega_1)/(k^2S_{\eta\eta}^+)\\ &S_2 = k_{22}^2S_{\eta\eta}(2\omega_2)/(k^2S_{\eta\eta}^+), \quad S_3 = S_{\eta\eta}^-/S_{\eta\eta}^+, \quad b = k^2S_{\eta\eta}^+(S_0 - 2)/32\end{aligned} \tag{53}$$

and substituting for $\bar{\beta}_i$ into the formula of Eq. (29), the stability boundaries corresponding to $\lambda = 0$ can be obtained from the following transcendental equation:

$$\bar{\beta}_1 e^{-\bar{\beta}_1} = \bar{\beta}_2 e^{-\bar{\beta}_2} \tag{54}$$

By choosing different values for the non-dimensional parameter β_1^* and the spectral density ratios S_1, S_2, S_3 and solving the transcendental equation (54) for $\bar{\beta}_2$, stability boundaries in terms of β_1^* and β_2^* can be obtained.

5.3. Follower force case

For the follower force case, the flexural and torsional equations of motion for a viscoelastic beam under central force and end moments are given as follows:

$$\begin{aligned} &EI_y(1-\mathbf{R})\frac{\partial^4 u}{\partial z^4} + \frac{\partial^2 (M_x \psi)}{\partial z^2} + P\psi_m \delta\left(z - \frac{1}{2}L\right) + m\frac{\partial^2 u}{\partial t_1^2} + D_u \frac{\partial u}{\partial t_1} = 0 \\ &-GJ(1-\mathbf{R})\frac{\partial^2 \psi}{\partial z^2} + M_x \frac{\partial^2 u}{\partial z^2} + mr^2 \frac{\partial^2 \psi}{\partial t_1^2} + D_\psi \frac{\partial \psi}{\partial t_1} = 0 \end{aligned} \tag{55}$$

where $\psi_m = \psi(L/2, t_1)$ is the value of the angle of twist $\psi(z, t_1)$ at $Z = L/2$ and $\delta(z - L/2)$ is the Dirac delta function centered at $L/2$. Substituting for u and ψ from Eq. (44) into the equations of motion (55), employing the Galerkin method and introducing the non-dimensional time scale $t = vt_1$, the following non-dimensional system of equations can be obtained.

$$\begin{aligned} &\ddot{q}_1 + \frac{L^2 D_u}{\pi^2\sqrt{EI_y m}}\dot{q}_1 + (1-\mathbf{R})q_1 + \frac{\gamma_{02}}{c^*}q_2 + \frac{1}{c^*}\left(\frac{12-\pi^2}{4+\pi^2}\xi_1 - \xi_2\right)q_2 = 0 \\ &\ddot{q}_2 + \frac{L^2 D_\psi}{\pi^2 r^2\sqrt{EI_y m}}\dot{q}_2 + \frac{GJL^2}{\pi^2 r^2 EI_y}(1-\mathbf{R})q_2 - c^*\gamma_{01}q_1 - c^*(\xi_1+\xi_2)q_1 = 0 \end{aligned} \tag{56}$$

Upon using the transformation

$$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \to \begin{bmatrix} 1 & -\gamma_{02}/(c^*(\omega_1^2 - \omega_2^2)) \\ -c^*(1-\omega_1^2)/\gamma_{02} & (1-\omega_2^2)/(\omega_1^2 - \omega_2^2) \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \tag{57}$$

where

$$\begin{aligned} &\omega_{1,2}^2 = \frac{1}{2}\left(1 + L^2 GJ/(\pi^2 r^2 EI_y) \mp \sqrt{(1 - L^2 GJ/(\pi^2 r^2 EI_y))^2 - 4\gamma_{01}\gamma_{02}}\right) \\ &\gamma_{02} = L^2((12-\pi^2)P_s L - 8\pi^2 M_s)/(8r\pi^4 EI_y) \end{aligned} \tag{58}$$

and pre-multiplying by the inverse of the transformation matrix of Eq. (57), non-dimensional equations of motion of the same form as Eq. (1) can be obtained. Without loss of generality it is always possible in the follower force case to scale $k_{11} = k_{22} = 1$ and $k_{12} = -k_{21} = 1$. To facilitate a suitable coordinate scaling, the parameter c^* is now chosen such that $c^* = 1$. The non-dimensional parameters β_{11}, β_{22}, e_{11}, e_{22} and γ_{01} and the ergodic processes ξ_1 and ξ_2 are of the same form as those for the non-follower force case. The constants coefficients c_{ijm}, $i, j, m = 1, 2$, which measure the contribution of the ergodic processes ξ_1 and ξ_2 to the multiplicative processes η_{ij}, $i, j = 1, 2$, are now obtained as

$$
\begin{aligned}
c_{111} &= \frac{(12-\pi^2)(1-\omega_1^2)(1-\omega_2^2)+(4+\pi^2)\gamma_{02}^2}{(4+\pi^2)\omega_1\gamma_{02}(\omega_2^2-\omega_1^2)} \\
c_{112} &= \frac{\gamma_{02}^2-(1-\omega_1^2)(1-\omega_2^2)}{\omega_1\gamma_{02}(\omega_2^2-\omega_1^2)} \\
c_{121} &= \frac{(12-\pi^2)(1-\omega_2^2)^2+(4+\pi^2)\gamma_{02}^2}{(4+\pi^2)\omega_1(\omega_2^2-\omega_1^2)^2} \\
c_{122} &= \frac{\gamma_{02}^2-(1-\omega_2^2)^2}{\omega_1(\omega_2^2-\omega_1^2)^2} \\
c_{211} &= \frac{(12-\pi^2)(1-\omega_1^2)^2+(4+\pi^2)\gamma_{02}^2}{(4+\pi^2)\omega_2\gamma_{02}^2} \\
c_{212} &= \frac{\gamma_{02}^2-(1-\omega_1^2)^2}{\omega_2\gamma_{02}^2}, \quad c_{221} = -\frac{\omega_1}{\omega_2}c_{111}, \quad c_{222} = -\frac{\omega_1}{\omega_2}c_{112}
\end{aligned}
\tag{59}
$$

In the same manner as for the non-follower force case, the transformation matrix of Eq. (57) is constructed from the eigenvectors of the stiffness matrix of the elastic system of equations (56). The frequencies $\omega_{1,2}$ are real if

(i) $\gamma_{02} < 0$ and $\frac{L^2GJ}{\pi^2r^2EI_y} > -\gamma_{01}\gamma_{02}$.
(ii) $\gamma_{02} > 0$ and $\frac{L^2GJ}{\pi^2r^2EI_y} < 1-2(\gamma_{01}\gamma_{02})^{1/2}$, or $\frac{L^2GJ}{\pi^2r^2EI_y} > 1+2(\gamma_{01}\gamma_{02})^{1/2}$.

Condition (i) implies that

$$
\left(\frac{12-\pi^2}{4+\pi^2}\frac{P_s}{P_{\rm cr}}-\frac{M_s}{M_{\rm cr}}\right) < 0 \tag{60}
$$

and

$$
\left(\frac{P_s}{P_{\rm cr}}+\frac{M_s}{M_{\rm cr}}\right)\left(\frac{M_s}{M_{\rm cr}}+\frac{12-\pi^2}{4+\pi^2}\frac{P_s}{P_{\rm cr}}\right) < 1 \tag{61}
$$

whereas condition (ii) implies that

$$
\left(\frac{12-\pi^2}{4+\pi^2}\frac{P_s}{P_{\rm cr}}-\frac{M_s}{M_{\rm cr}}\right) > 0 \tag{62}
$$

and

$$
\left(\frac{P_s}{P_{\rm cr}}+\frac{M_s}{M_{\rm cr}}\right)\left(\frac{12-\pi^2}{4+\pi^2}\frac{P_s}{P_{\rm cr}}-\frac{M_s}{M_{\rm cr}}\right) < \frac{1}{4}\left(\frac{\pi r}{L}\sqrt{\frac{EI_y}{GJ}}-\frac{L}{\pi r}\sqrt{\frac{GJ}{EI_y}}\right)^2 \tag{63}
$$

$P_{\rm cr}$ and $M_{\rm cr}$ are the static buckling loads in the non-follower force case. From condition (ii) and under the action of the central force only, i.e. by setting $M_s = 0$, the value of P_s at which dynamic buckling can occur is obtained as

$$
P_s^* = \frac{4\pi^2|L^2GJ-\pi^2r^2EI_y|}{rL^3(12-\pi^2)^{1/2}(4+\pi^2)^{1/2}} \tag{64}
$$

This value agrees with that obtained earlier by Ariaratnam et al. (1992). By defining the non-dimensional parameters

$$\bar{\beta}_i = \frac{1}{4}(-\beta_i^* + 8S_i - 8S_3)\eta_0, \quad i = 1,2 \tag{65}$$

where

$$\beta_i^* = \frac{32}{(S^+_{\eta_{12}\eta_{12}} + S^+_{\eta_{21}\eta_{21}})}(2\beta_{ii} + \omega_i e_{ii} R_s(\omega_i))$$

and

$$\eta_0 = \begin{cases} \frac{1}{2(4S_4^2 - S_0^2)^{1/2}} \cos^{-1}\left(\frac{S_0^2 - 2S_4^2}{2S_4^2}\right) & \Delta > 0 \\ \frac{1}{2(S_0^2 - 4S_4^2)^{1/2}} \cosh^{-1}\left(\frac{S_0^2 - 2S_4^2}{2S_4^2}\right) & \Delta < 0 \\ \frac{1}{S_0} & \Delta = 0 \end{cases} \tag{66}$$

$$\begin{aligned} &S_0 = S_1 + S_2 + 2S_3, \quad S_1 = S_{\eta_{11}\eta_{11}}(2\omega_1)/(S^+_{\eta_{12}\eta_{12}} + S^+_{\eta_{21}\eta_{21}}) \\ &S_2 = S_{\eta_{22}\eta_{22}}(2\omega_2)/(S^+_{\eta_{12}\eta_{12}} + S^+_{\eta_{21}\eta_{21}}), \quad S_3 = S^-_{\eta_{12}\eta_{21}}/(S^+_{\eta_{12}\eta_{12}} + S^+_{\eta_{21}\eta_{21}}) \\ &S_4 = (S^+_{\eta_{12}\eta_{12}} S^+_{\eta_{21}\eta_{21}})^{1/2}/(S^+_{\eta_{12}\eta_{12}} + S^+_{\eta_{21}\eta_{21}}), \quad \Delta = 4(4S_4^2 - S_0^2)/(1+S_0)^2 \end{aligned} \tag{67}$$

and substituting for $\bar{\beta}_i$ into the formula of Eq. (27) and following the same procedure as that for the non-follower force case, stability boundaries in terms of β_1^* and β_2^* can be obtained.

6. Numerical results and discussion

Modern aerospace and other structures are often constructed of materials that are more nearly viscoelastic than elastic. Viscoelastic behaviour is observed in a number of materials which are extremely important in applications using polymers and plastics, composite materials, concrete, soil, road construction and building materials. For viscoelastic materials stress is not an instantaneous function of strain but depends on the past time history of strain and vice versa. The constitutive relations that describe such hereditary materials are usually integral relations possessing a relaxation kernel function rather than algebraic equations of ordinary elasticity. Consequently, the governing equations of motion that describe the dynamic response of viscoelastic systems are integro-differential equations.

A viscoelastic material having elastic bulk response and with a constitutive relation described as the three parameter standard solid is considered for the present analysis. The viscoelastic model for standard solid material is constructed by taking a Maxwell element in parallel with a spring or by taking a Kelvin element in series with a spring. The constitutive relation for the deviatoric parts of the stress and strain tensors of this material, $\hat{s}$ and $\hat{e}$, respectively, can be shown to be given by the following differential equation:

$$\hat{s} + p_1\dot{\hat{s}} = q_0\hat{e} + q_1\dot{\hat{e}} \tag{68}$$

where p_1, q_0 and q_1 are positive constants with a system physical necessity $q_1 > p_1 q_0$. By solving the differential equation (68), the following relation can be obtained:

$$\hat{s} = \frac{q_1}{p_1}\left(\hat{e} - \frac{q_1 - q_0 p_1}{q_1 p_1}\int_0^t e^{-\frac{1}{p_1}(t-\tau)}\hat{e}(\tau)\,d\tau\right) \tag{69}$$

Eq. (69) is of the same form as that of Eq. (41) with $2G = q_1/p_1$ and a relaxation kernel given by $R(t) = \frac{\chi_i}{T_i}e^{-\frac{t}{T_i}}$ where χ_i and T_i are the non-dimensional characteristic viscosity and relaxation time, respectively, which are obtained as $\chi_i = 1 - q_0 p_1/q_1$ and $T_i = p_1$.

A more sophisticated viscoelastic model can be constructed by taking a Kelvin element in series with a spring and a dashpot. The constitutive relation for the deviatoric parts of the stress and strain tensors, $\hat{s}$ and $\hat{e}$, of such a model is described by the following differential equation:

$$\hat{s} + p_1\dot{\hat{s}} + p_2\ddot{\hat{s}} = q_1\dot{\hat{e}} + q_2\ddot{\hat{e}} \tag{70}$$

where p_1, p_2, q_1 and q_2 are positive constants with a system physical necessity $p_1^2 > 4p_2$, $p_1q_1 > q_2$, and $p_1q_1q_2 > p_2q_1^2 + q_2^2$. By solving Eq. (70), the following relation can be obtained:

$$\hat{s} = \frac{q_2}{p_2}\left(\hat{e} - \sum_{i=1}^{2}\frac{\chi_i}{T_i}\int_0^t \mathrm{e}^{-\frac{1}{T_i}(t-\tau)}\hat{e}(\tau)\,\mathrm{d}\tau\right) \tag{71}$$

Eq. (71) is of the same form as that of Eq. (41) with $2G = q_2/p_2$ and a relaxation kernel given by $R(t) = \sum_{i=1}^{2}\frac{\chi_i}{T_i}\mathrm{e}^{-\frac{t}{T_i}}$ where

$$\begin{aligned}
\chi_1 &= \frac{p_2[(p_1q_2 - p_2q_1)\alpha_1 - q_2]}{q_2\alpha_1\sqrt{p_1^2 - 4p_2}}, \quad T_1 = \frac{1}{\alpha_1} \\
\chi_2 &= \frac{p_2[(p_2q_1 - p_1q_2)\alpha_2 + q_2]}{q_2\alpha_2\sqrt{p_1^2 - 4p_2}}, \quad T_2 = \frac{1}{\alpha_2} \\
\alpha_{1,2} &= \frac{p_1 \pm \sqrt{p_1^2 - 4p_2}}{2p_2}
\end{aligned} \tag{72}$$

For some viscoelastic models, the viscoelastic relaxation operator **R** is expressed in a more rigorous form other than the simple integral form of Eq. (2). To show this, consider a four parameter viscoelastic model constructed by taking two Kelvin element in series. The constitutive relation for the deviatoric parts of the stress and strain tensors of such a model is obtained as

$$\hat{s} + p_1\dot{\hat{s}} = q_0\hat{e} + q_1\dot{\hat{e}} + q_2\ddot{\hat{e}} \tag{73}$$

where p_1, q_0, q_1 and q_2 are positive constants with a system physical necessity $q_1 > q_0p_1$, $q_1^2 > 4q_0q_2$, $q_1p_1 > q_0p_1^2 + q_2$. The solution of Eq. (73) is obtained as the following:

$$\hat{s} = 2G\left(\hat{e} - \frac{\chi_i}{T_i}\int_0^t \mathrm{e}^{-\frac{1}{T_i}(t-\tau)}\hat{e}(\tau)\,\mathrm{d}\tau\right) + \frac{q_2}{p_1}\dot{\hat{e}} \tag{74}$$

where $2G = (q_1p_1 - q_2)/p_1^2$, $\chi_i = 1 - q_0p_1^2/(q_1p_1 - q_2)$ and $T_i = p_1$. The solution of Eq. (74) is of the same form as that of Eq. (41) with the viscoelastic relaxation operator given by

$$\mathbf{R}[\psi(t)] = \int_0^t R(t-\tau)\psi(\tau)\,\mathrm{d}\tau - (q_2p_1/(q_1p_1 - q_2))\frac{\mathrm{d}\psi(t)}{\mathrm{d}t} \tag{75}$$

More sophisticated viscoelastic models can be constructed by taking a combination of Maxwell and Kelvin elements taken in parallel or in series. Consequently, more rigorous forms of the viscoelastic relaxation operator, **R**, are required to analyze the viscoelastic system under consideration.

The motivation for the present study stems from the investigation of flexural–torsional instability of a deep rectangular viscoelastic beam subjected to stochastically fluctuating central transverse load and end moments applied simultaneously. For many cases, the central transverse force and end moments are principally static loads. Even the loads are basically static, in some situations, it may be more realistic to allow for random perturbations. The application of static loading subjected to random perturbation leads to dynamical systems with a coupled stiffness matrix. Previous works dealt with the investigation of elastic beams that are subjected to either stochastic end moments or central transverse load applied separately and neglected to consider for the static parts. Stochastic moment stability, in the mean and mean square, of an elastic rectangular beam subjected to random end moments with zero mean was investigated by Ariaratnam and Srikantaiah (1978) and sufficient stability conditions were obtained. Ariaratnam et al. (1992) studied the stochastic stability of coupled linear systems and applied the obtained results to investigate the stochastic stability of an elastic beam subjected to central random load with zero mean by calculating the largest Lyapunov exponent. The almost-sure stochastic stability of a viscoelastic beam subjected to both central transverse force and end moments applied simultaneously appears not to have been investigated before. A broad class of non-gyroscopic viscoelastic systems is treated in the present analysis and a general formulation is performed so that the case of a viscoelastic beam under combined loading can be considered as an application.

Many researchers investigated the stability of viscoelastic dynamical systems, such as Touati and Cederbaum (1994), Potapov and Bonder (1996) and Potapov (1997) by introducing the viscoelastic terms as new variables and augmenting the dynamical systems under consideration. By augmenting the viscoelastic system, it will be very difficult if not impossible, using Khasminiskii's techniques, to investigate the almost-sure stochastic stability analytically for two and higher degrees of freedom systems. Also by augmenting the viscoelastic dynamical systems, more computational effort and time are needed to investigate the stability numerically. By employing the method of Larianov's (1969) in the present investigation, the viscoelastic terms are averaged as deterministic terms and the almost-sure stochastic stability of the viscoelastic dynamical systems is investigated without the need to augment the system. In the present study, the formulation is given in a more general form so that the combined loading of end moments and follower transverse force case can be included. More general new results, for the extremely different loading conditions of Case 1, are derived in Eq. (27). The results obtained in the earlier works of Ariaratnam and Srikantaiah (1978) and Ariaratnam et al. (1992) can therefore, be deduced as special results from the present study. When the method of stochastic averaging is used in the first approximation, only values of the excitation spectrum at the frequencies $\omega = 2\omega_1, 2\omega_2, \omega_1 \pm \omega_2$ have effect on the stability. If higher order approximations of the stochastic averaging method are used, stability may be affected by values of the excitation spectrum at other multiples or fractions of the natural frequencies.

The results obtained for the two-degrees of freedom system can be generalized to n-degrees of freedom system under certain conditions on the spectral density distribution of the parametric independent ergodic processes $\xi_1(t)$ and $\xi_2(t)$. For band-limited excitation, the spectral density is considered to be small everywhere when compared with those near the neighborhood of some frequency ω_0; thus, $S(\omega)$ is concentrated in a narrow bandwidth, $\omega_0 - \Delta\omega_0/2 < \omega < \omega_0 + \Delta\omega_0/2$, where $\Delta\omega_0 \ll \omega_0$. For such a process with a spectral density $S(\omega) = O(\epsilon), 0 < |\epsilon| \ll 1$, the correlation time τ_c is $O(1/\Delta\omega_0)$, while the relaxation time τ_r is $O(1/\epsilon)$. Hence if $\Delta\omega_0 \gg \epsilon$, then $\tau_c \ll \tau_r$ and the Markov process approximation made in the previous sections will remain valid. By considering band-limited excitation processes, the largest Lyapunov exponents for the n-degrees of freedom system when ω_0 lies in the neighborhood of $2\omega_1, 2\omega_2$, and $\omega_1 \pm \omega_2$ can be deduced from those obtained for the two-degrees of freedom system.

For a rectangular beam, the parameters J and r are given as $J = wh^3/3$ and $r = h/\sqrt{12}$. By using the relation $G = E/(1+v)$, where v is the Poisson's ratio, and taking $v = 1/3$, the non-dimensional frequencies ω_1 and ω_2 can be obtained in terms of the ratio L/h. The non-dimensional parameters γ_{01} and γ_{02}, the terms k_{11}, k_{22} and k for the non-follower force case and the coefficients $c_{ij\ell}$, $i, j, \ell = 1, 2$ for the follower force case, can be expressed in terms of the ratios L/h, $P_s/(Ewh)$ and $M_s/(Ewh^2)$. For numerical illustrations, a viscoelastic material, known as the three parameter standard solid, with $\chi_i = 0.9$, $T_i = 2$ and the non-dimensional parameters $L/h = 10$, $P_s/(Ewh) = 0.01$ and $M_s/(Ewh^2) = 0.02$ are considered in the present analysis. The non-dimensional static loading parameters can be calculated as $\gamma_{01} = 15.822$ and

$\gamma_{02} = -7.2872$. Since the stochastic averaging method is used in the first approximation, only spectral densities at the frequencies $\omega = 2\omega_1, 2\omega_2, \omega_1 \pm \omega_2$ have an effect on stability. It is also found that the one-sided Fourier sine and cosine transforms of the viscoelastic relaxation function, $R_s(\omega)$ and $R_c(\omega)$, evaluated at the natural frequencies of the system play the role of effective viscous damping and additional elastic

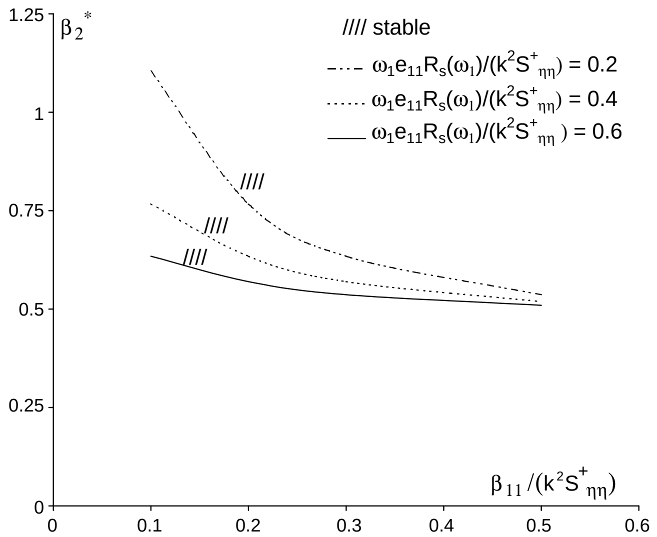

Fig. 2. Effect of the viscoelastic term $\omega_1 e_{11} R_s(\omega_1)$ on stability boundaries of a viscoelastic beam under non-follower force with $S_1 = 0.75$, $S_2 = 0.5$, $S_3 = 0.25$.

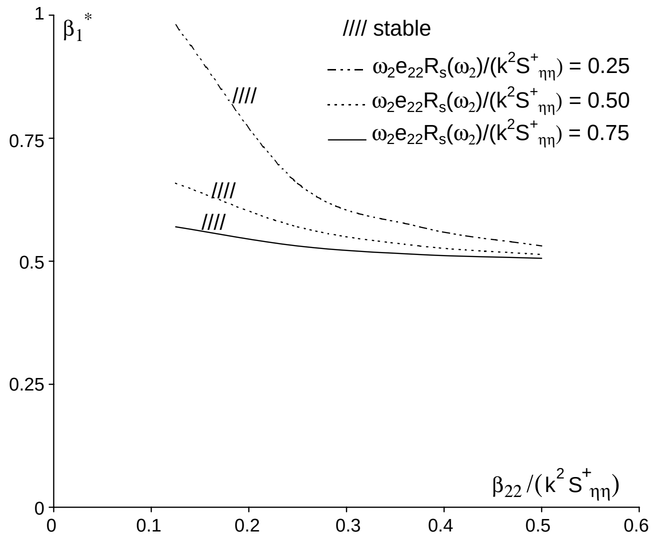

Fig. 3. Effect of the viscoelastic term $\omega_2 e_{22} R_s(\omega_2)$ on stability boundaries of a viscoelastic beam under non-follower force with $S_1 = 0.75$, $S_2 = 0.5$, $S_3 = 0.25$.

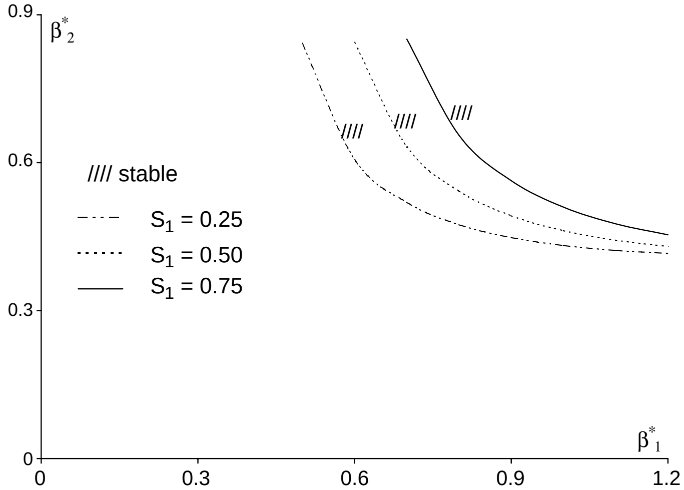

Fig. 4. Effect of the spectral density ratio S_1 on stability boundaries of a viscoelastic beam under non-follower force with $S_2 = 0.5$, $S_3 = 0.75$.

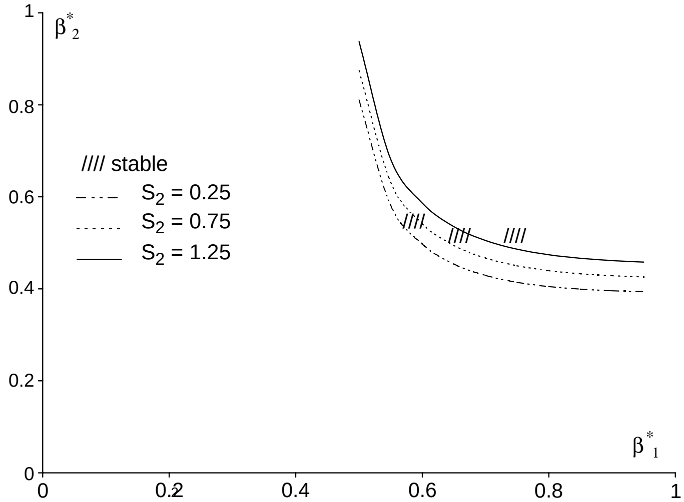

Fig. 5. Effect of the spectral density ratio S_2 on stability boundaries of a viscoelastic beam under non-follower force with $S_1 = 0.25$, $S_3 = 0.75$.

stiffness. The viscoelastic terms $\omega_i e_{ii} R_s(\omega_i)$, $i = 1, 2$, are observed to have a stabilizing effect on the system considered as can be seen from Figs. 2 and 3. For the non-follower force case, it can be seen from Figs. 4–6, that the spectral density ratios S_1, S_2 and S_3 have a destabilizing effect. By considering this result and using the definition of the corresponding spectral density ratios, it can be inferred that the spectral densities $S_{\eta\eta}(2\omega_1)$, $S_{\eta\eta}(2\omega_2)$ and $S_{\eta\eta}(\omega_1 + \omega_2)$ have a destabilizing effect whereas $S_{\eta\eta}(\omega_1 - \omega_2)$ has a stabilizing effect. Defining the non-dimensional parameters $\alpha_i = -8\lambda_i\eta_0/k^2$, $i = 1, 2$, the spectral density of the non-follower white noise central force has a destabilizing effect as can be seen from Fig. 7. For the follower force case, it

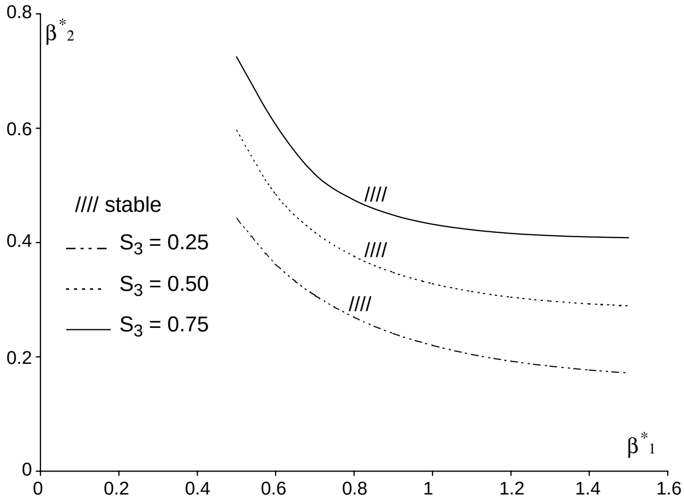

Fig. 6. Effect of the spectral density ratio S_3 on stability boundaries of a viscoelastic beam under non-follower force with $S_1 = 0.25$, $S_2 = 0.5$.

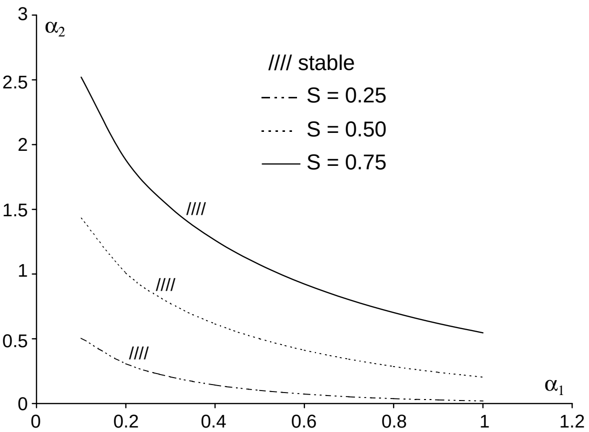

Fig. 7. Effect of the spectral density ratio S on stability boundaries of a viscoelastic beam under white noise non-follower force.

can be inferred from Figs. 8–11 that the densities $S_{\eta_{11}\eta_{11}}(2\omega_1)$, $S_{\eta_{22}\eta_{22}}(2\omega_2)$ and $S_{\eta_{12}\eta_{21}}(\omega_1 - \omega_2)$ have a destabilizing effect, whereas the densities $S_{\eta_{12}\eta_{21}}(\omega_1 + \omega_2)$, $S_{\eta_{12}\eta_{12}}(\omega_1 - \omega_2)$, $S_{\eta_{12}\eta_{12}}(\omega_1 + \omega_2)$, $S_{\eta_{21}\eta_{21}}(\omega_1 - \omega_2)$, and $S_{\eta_{21}\eta_{21}}(\omega_1 + \omega_2)$ have a stabilizing effect. Again stability boundaries in the space of the non-dimensional parameters $\alpha_i = 8\lambda_i\eta_0$, $i = 1, 2$, are obtained for the follower force case. The spectral density of the follower white noise central force has a destabilizing effect as can be seen from Fig. 12.

The ergodic stochastic processes $\xi_1(t)$ and $\xi_2(t)$ are obtained in terms of the central transverse force $P(t)$ and end moments $M(t)$, respectively. The spectral densities of the multiplicative processes $S_{\eta_{ij}\eta_{rs}}$, $i, j, r, s = 1, 2$, using Eq. (5), can be given in terms of the spectral densities of the independent processes $\xi_1(t)$ and $\xi_2(t)$ by the relation: $S_{\eta_{ij}\eta_{rs}}(\omega) = c_{ij1}c_{rs1}S_{\xi_1\xi_1}(\omega) + c_{ij2}c_{rs2}S_{\xi_2\xi_2}(\omega)$. Knowing the effect of the spectral

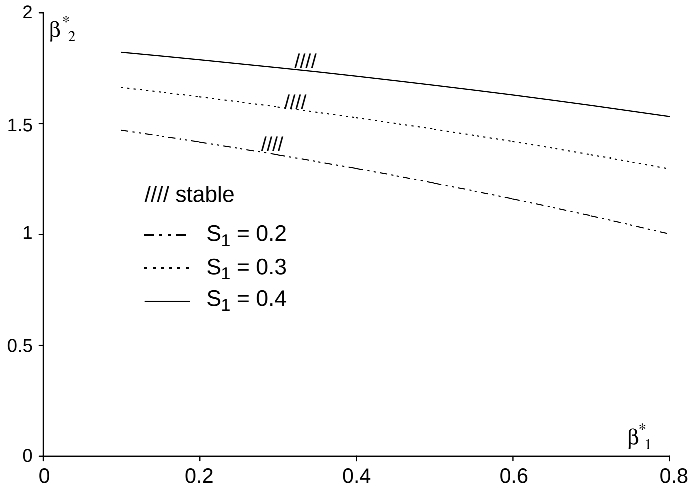

Fig. 8. Effect of the spectral density ratio S_1 on stability boundaries of a viscoelastic beam under follower force with $S_2 = 0.5$, $S_3 = -0.25$, $S_4 = 0.2$.

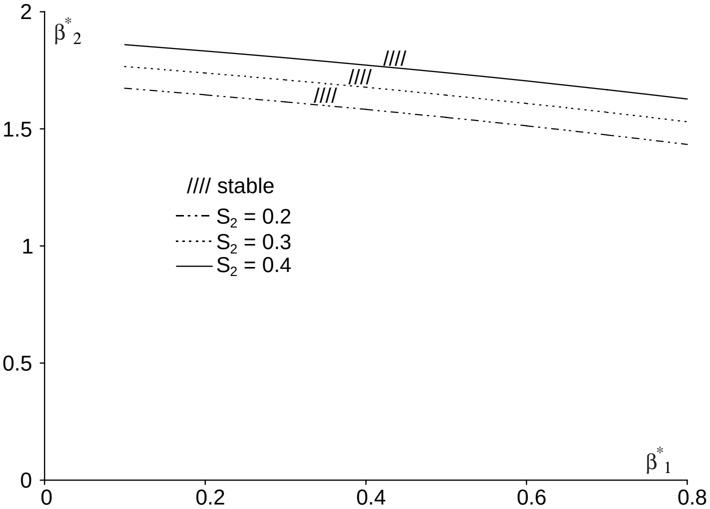

Fig. 9. Effect of the spectral density ratio S_2 on stability boundaries of a viscoelastic beam under follower force with $S_1 = 0.5$, $S_3 = -0.25$, $S_4 = 0.2$.

densities $S_{\eta_{ij}\eta_{rs}}(\omega)$ on stability, the effect of the spectral densities $S_{\xi_1\xi_1}(\omega)$ and $S_{\xi_2\xi_2}(\omega)$ is determined by evaluating the coefficients $c_{ij\ell}c_{rs\ell}$, $i,j,r,s,\ell = 1,2$. For the non-follower force case it is found that $c_{ij\ell} = 1$, $i,j,\ell = 1,2$ and thus, it can be inferred that the spectral densities of the ergodic processes $\xi_1(t)$ and $\xi_2(t)$ have a destabilizing effect at the frequencies $2\omega_1$, $2\omega_2$ and $\omega_1 + \omega_2$ and a stabilizing effect at the frequency $\omega_1 - \omega_2$. For the follower force case, the coefficients $c_{ij\ell}$, $i,j,\ell = 1,2$, are given in terms of the parameters ω_1, ω_2 and γ_{02}. This implies that, for the follower force case, the effect of the spectral densities of the ergodic

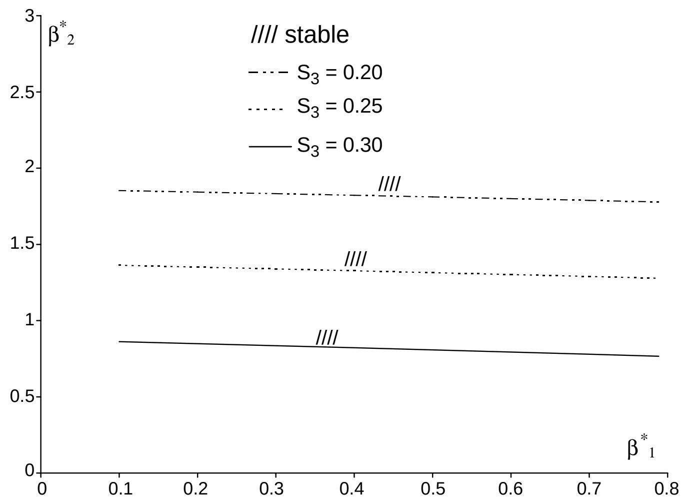

Fig. 10. Effect of the spectral density ratio S_3 on stability boundaries of a viscoelastic beam under follower force with $S_1 = 6$, $S_2 = 5$, $S_4 = 0.2$.

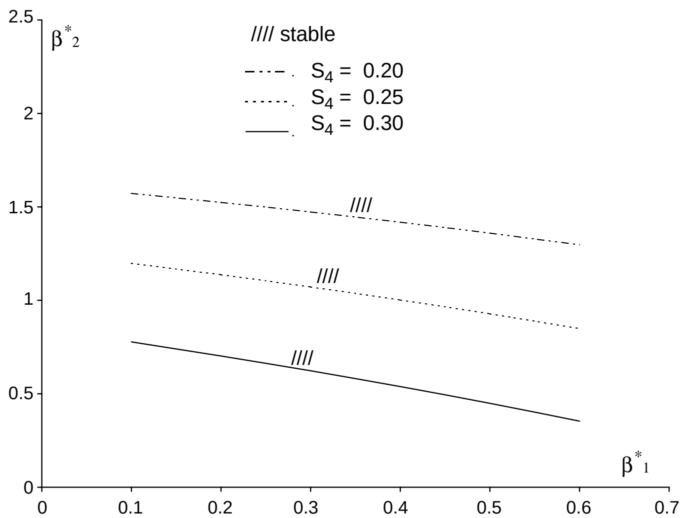

Fig. 11. Effect of the spectral density ratio S_4 on stability boundaries of a viscoelastic beam under follower force with $S_1 = 0.25$, $S_2 = 0.5$, $S_3 = -0.25$.

processes $\xi_1(t)$ and $\xi_2(t)$ on stability is determined in part by the ratios L/h, $P_s/(Ewh)$ and $M_s/(Ewh^2)$. The results obtained in the present analysis can be expressed in terms of the dimensional spectral densities of the excitation processes $\tilde{\eta}_{ij}(t_1)$, $i, j = 1, 2$ where $t = vt_1$. The dimensional spectral densities are given by the following relations:

$$S_{\tilde{\eta}_{ij}\tilde{\eta}_{rs}}(v\omega) = \frac{1}{v} S_{\eta_{ij}\eta_{rs}}(\omega), \quad i, j, r, s = 1, 2, \ \omega = 2\omega_1, 2\omega_2, \omega_1 \pm \omega_2 \tag{76}$$

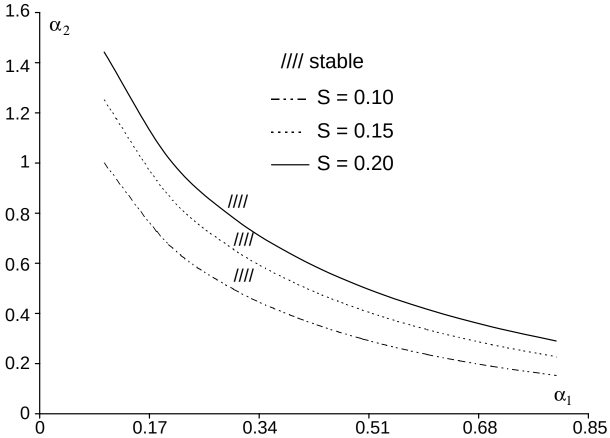

Fig. 12. Effect of the spectral density ratio S on stability boundaries of a viscoelastic beam under white noise follower force.

7. Conclusions

The stability of a viscoelastic non-gyroscopic system described by a stochastic integro-differential equation was investigated. The system was parametrically excited by a force given by a linear combination of ergodic stochastic processes of small intensity and a short correlation time. Explicit expressions for the largest Lyapunov exponent as an almost-sure stability indicator, valid in the first approximation, were obtained by making use of the stochastic averaging method for the non-viscoelastic terms together with Khasminskii's technique. The integral term arising from the viscoelastic effect was averaged by employing Larianov's method. The obtained results were applied to investigate the stability of a narrow and deep rectangular viscoelastic beam under random transverse central load and end moments applied simultaneously. Both cases of follower and non-follower central loads were considered. The effect of the excitation spectrum at the frequencies, $\omega = 2\omega_1, 2\omega_2, \omega_1 \pm \omega_2$, on stability was investigated and stability boundaries in the space of non-dimensional parameters, given in terms of the system parameters were obtained.

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